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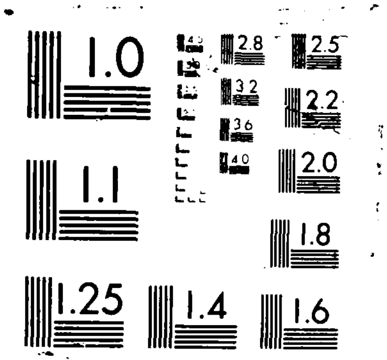
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CMS Technical Summary Report #88-9

COMPUTATION OF THE EIGENVALUES FOR PERTURBATIONS OF POISEUILLE FLOW USING A TWO-POINT BOUNDARY VALUE METHOD

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October 1987

(Received October 8, 1987)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

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ABSTRACT

We compute the decay rates for stationary perturbations of Poiseuille flow in channels and pipes. The decay rates are found by solving eigenvalue problems of ordinary differential equations, where the eigenvalues give the rate of decay for the perturbation. A two-point boundary value method is used to compute the eigenvalues yielding efficient and accurate calculations. For the channel flow problem, the results are in agreement with previous calculations, (e.g. [3], [4], [5], [7], [12]) however, the problem of determining the rate of decay for a fluid motion in a pipe has not been considered before. ~~We prove that~~ for the Stokes problem in a pipe the eigenvalues, governing the rate of decay, are complex. We carry out computations for small and moderate Reynolds numbers, also high Reynolds number computations were done to show the effectiveness of this method.

AMS(MOS) Subject Classifications: 65L15, 76D07

Key Words: Navier-Stokes, eigenvalue problem, Poiseuille flow, PASVA3, Reynolds number, asymptotic

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Supported by the U. S. Army Research Office under Contract No. DAAL03-87-K-0028.

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1. Introduction

In this paper we are concerned with the eigenvalue problem which governs the rate of decay for a stationary perturbation of Poiseuille flow. We consider two-dimensional viscous motions in channels and axi-symmetric viscous flow in pipes. We assume that the difference between the base flow and Poiseuille flow decays exponentially downstream (or upstream). It is then possible to seek solutions to the Navier-Stokes equations, far downstream (or upstream), that are a perturbation to the Poiseuille profile and that decay exponentially in the axial direction. The equations can then be linearized yielding an ordinary differential eigenvalue system where the eigenvalues determine the rate of decay for the stationary perturbation.

By use of the stream function formulation, it is possible to reduce the eigenvalue system to a single fourth-order differential eigenvalue equation for the decay of the stationary perturbation. In the two-dimensional case this differential equation is very similar to the Orr-Sommerfeld stability equation. Results regarding the computation of these eigenvalues, in the case of channel flow, have been presented previously, e.g. [3],[4],[5],[7],[12], however when the fluid motion is considered axially symmetric, the problem of determining the rate of decay for the stationary perturbation has not been considered before.

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To compute the eigenvalues several methods have been used, for example, a spectral method by Bramley [3], and Bramley and Dennis [4], an initial value method by Bramley and Dennis [5], a singular perturbation method by Wilson [12], etc. All of these works have dealt with the two-dimensional channel flow problem, and with the exception of Bramley [3] all of them have considered the Navier-Stokes equations in the stream function formulation. The work of Bramley and Dennis [4] appears to be the most complete among all others mentioned above. They used an extension of the Orszag's spectral method presented in [9]. In [3] Bramley used the same method to compute the eigenvalues using the primitive formulation for the Navier-Stokes equations, i.e., he used the velocity field and the pressure instead of the stream function. However, the results of Bramley [3] are not in complete agreement with [4]. This disagreement may be the result of using the wrong number of boundary conditions. There are two disadvantages reported in [4] with respect to spectral methods, these are the computation of spurious eigenvalues and the loss of accuracy for high Reynolds numbers computations, especially for eigenvalues with negative real part. In [5] Bramley and Dennis used an initial value method to compute the eigenvalues. They said that using this method it is possible to overcome some of the difficulties of spectral method. However, their method computes only eigenvalues, and not eigenfunctions, and only one at a time.

In this paper we compute the eigenvalues for a perturbation of Poiseuille flow, in a channel and a pipe, using a more accurate and efficient method. This method can be described in two steps, first, we transform the eigenvalue problem into an equivalent two-point boundary value system, then we numerically solve this system using the two-point

boundary value solver DVCPR from the IMSL library. For the two-dimensional problem our results are in full agreement with the previous work of Bramley and Dennis [4], for small and moderate Reynolds numbers, and with the asymptotic prediction of Wilson [12], for eigenvalues approaching zero at high Reynolds numbers. The numerical results presented in this paper show the efficiency of the calculations and the superiority of our method with respect to the other methods mentioned above. These eigenvalues are important for the derivation of boundary conditions, this problem was considered in [1].

2. Channel Flow

Given a semi-infinite channel, the incompressible Navier-Stokes equations are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{R}\nabla^2 u \ , \tag{2.1a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{R}\nabla^2 v \ , \tag{2.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \tag{2.1c}$$

with boundary conditions

$$u = v = 0 \quad \text{at} \quad y = \pm 1 \ , \tag{2.1d}$$

$$u = F_1(y) \ , \quad v = F_2(y) \quad \text{at} \quad x = 0 \ , \tag{2.1e}$$

where F_i is a profile satisfying $F_i(\pm 1) = 0$. Finally we have the regularity condition

$$(u, v) \rightarrow (\hat{u}, 0) \quad \text{as} \quad x \rightarrow \infty \ . \tag{2.1f}$$

Here R represents the Reynolds number, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\hat{u} = \frac{3}{2}(1 - y^2)$ is the Poiseuille parabolic profile.

Since we are only interested in solutions of this system which decay exponentially, far downstream we seek an asymptotic solution to (2.1) in the form

$$u\,(x,y) = \frac{3}{2}(1-y^2) + W_1(y)\exp(-\lambda x) \ , \tag{2.2a}$$

$$v\,(x,y) = W_2(y)\exp(-\lambda x) \ , \tag{2.2b}$$

$$p\,(x,y) = -3R^{-1}x + R^{-1}\bar{q}(y)\exp(-\lambda x) + \bar{C} \ , \tag{2.2c}$$

where $\bar{C}$ is an arbitrary constant.

Substituting these expressions in equations (2.1a) to (2.1c), and neglecting nonlinear terms, we have that $(\lambda, W_1, W_2, \bar{q})$ satisfies the following eigenvalue system of ordinary differential equations,

$$\frac{d^2W_1}{dy^2} = -R\{\frac{3}{2}(1-y^2)\lambda W_1 + 3yW_2\} - \lambda\bar{q} - \lambda^2 W_1 \ , \tag{2.3a}$$

$$\frac{dW_2}{dy} = \lambda W_1 \ , \tag{2.3b}$$

$$\frac{d\bar{q}}{dy} = \frac{3}{2}R(1-y^2)\lambda W_2 + \lambda^2 W_2 - \lambda\frac{dW_1}{dy} \ , \tag{2.3c}$$

and boundary conditions

$$W_1(\pm 1) = W_2(\pm 1) = 0 \quad . \tag{2.3d}$$

We are interested in solving this two-point boundary value system to determine the rate of decay λ. For x large (i.e., downstream) the solution to (2.1) may be represented by

$$u\,(x,y) = \frac{3}{2}(1-y^2) + \sum_n W_{1,n}(y)\exp(-\lambda_n x) \ , \tag{2.4a}$$

$$v\,(x,y) = \sum_n W_{2,n}(y)\exp(-\lambda_n x) \ , \tag{2.4b}$$

and

$$q(x,y) = -3R^{-1}x + R^{-1}\sum_n \bar{q}_n(y)\exp(-\lambda_n x) + \bar{C} \ , \tag{2.4c}$$

$$Re(\lambda_n) \leq Re(\lambda_{n+1}) \ . \tag{2.4d}$$

The sum in (2.4a) to (2.4c) is taken over all the eigenvalues with positive real part (the eigenvalues with negative real part can be used to construct an asymptotic solution to (2.1) in a semi-infinite channel on the negative x-axis) and $(W_{1,n}, W_{2,n}, \bar{q}, \lambda_n)$ being a solution to the system (2.3).

We may eliminate the perturbed pressure $\bar{q}$ from the system (2.3) by differentiating (2.3a) with respect to y, multiplying equation (2.3a) by $-\lambda$, adding the resulting expressions and using (2.3b), then the system (2.3) is reduced into a single fourth-order differential equation in W_2 and λ very similar to the Orr-Sommerfeld equation (for simplicity we replace W_2 by W)

$$\frac{d^4W}{dy^4} + 2\lambda^2\frac{d^2W}{dy^2} + \lambda^4 W + \lambda R\{\frac{3}{2}(1-y^2)(\frac{d^2W}{dy^2} + \lambda^2 W) + 3W\} = 0 \ , \tag{2.5a}$$

and boundary conditions

$$W = \frac{dW}{dy} = 0 \ \text{ at } \ y = \pm 1 \ . \tag{2.5b}$$

We solved this equation in order to compare our method of solution with previous computations. Equation (2.5) is associated with the stream function formulation of the Navier-Stokes equations (see e.g. [4]).

When the Reynolds number R is equal to zero we may find an explicit solution to (2.5) and therefore to (2.3). This solution is given in terms of the Papkovitch-Fadle functions

for the biharmonic equation in a semi-infinite strip . For $R = 0$ equation (2.5) becomes

$$\frac{d^4W}{dy^4} + 2\lambda^2 \frac{d^2W}{dy^2} + \lambda^4 W = 0 \ , \tag{2.6}$$

and boundary conditions (2.5b). This equation is associated with the biharmonic equation when solutions of the form $w(x, y) = W(y)\exp(-\lambda x)$ are sought. There are two types of solutions for (2.6), one even and the other odd, which are given by

$$W^e(y) = (y-1)\sin(\lambda^e(y+1)) + (y+1)\sin(\lambda^e(y-1)) \ , \tag{2.7a}$$

and

$$W^o(y) = (y-1)\sin(\lambda^o(y+1)) - (y+1)\sin(\lambda^o(y-1)) \ , \tag{2.7b}$$

respectively. To satisfy the boundary conditions (2.5b) we need that the eigenvalues λ^e and λ^o satisfy the following transcendental equations,

$$\sin(2\lambda^e) + 2\lambda^e = 0 \ , \tag{2.8a}$$

and

$$\sin(2\lambda^o) - 2\lambda^o = 0 \ , \tag{2.8b}$$

respectively. We notice that if λ is an eigenvalue satisfying (2.8) so is $-\lambda$ and also $\pm\bar{\lambda}$, also the function

$$w(x, y) = \sum_n \alpha_n W_n(y)\exp(-\lambda_n x) \ , \tag{2.9}$$

where the sum is taken over the eigenvalues with positive real part, satisfies the biharmonic equation in a semi-infinite strip with transversal boundary conditions $w = \partial w/\partial y = 0$.

In section 4 we discuss a method to solve the eigenvalue problems (2.3) and (2.5), where the solution to the Stokes problem (2.6) plays an important role. Also in section 5

we present numerical results involving the computation of these eigenvalues for arbitrary Reynolds numbers.

3. Axially Symmetric Flow

In this section we consider the incompressible Navier-Stokes equations in cylindrical coordinates and dimensionless form, set in a semi-infinite pipe. These equations can be written as,

$$u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} + \frac{\partial p}{\partial r} = \frac{1}{R}\left(\nabla^2 u - \frac{u}{r^2}\right) , \tag{3.1a}$$

$$u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z} + \frac{\partial p}{\partial z} = \frac{1}{R}\nabla^2 v , \tag{3.1b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0 , \tag{3.1c}$$

and boundary conditions

$$u = v = 0 \text{ at } r = 1 . \tag{3.1d}$$

Since the fluid motion is symmetric the conditions at the center line are

$$u = \frac{\partial v}{\partial r} = 0 \text{ at } r = 0 , \tag{3.1e}$$

we also specify the entry condition

$$u = F_1(r) \ , v = F_2(r) \quad \text{at} \ z = 0 , \tag{3.1f}$$

finally we have the regularity condition.

$$(u, v) \to (0, \hat{v}) \quad \text{as} \ z \to \infty . \tag{3.1g}$$

Here $\nabla^2 = 1/r \ \partial/\partial r \ (r\partial/\partial r\ ) + \partial^2/\partial z^2$, and $\hat{v} = 1 - r^2$.

Similar to the channel flow problem we seek an asymptotic solution to (3.1) in the form,

$$u(r,z) = W_1(r)\exp(-\lambda z) \ , \tag{3.2a}$$

$$v(r,z) = 1 - r^2 + W_2(r)\exp(-\lambda z) \ , \tag{3.2b}$$

and

$$p(r,z) = -4R^{-1}z + R^{-1}\bar{q}(r)\exp(-\lambda z) + \bar{C} \ , \tag{3.2c}$$

with $\bar{C}$ an arbitrary constant.

Substituting these expressions in (3.1a) to (3.1c), and neglecting nonlinear terms, we obtain an eigenvalue system in W_1, W_2, $\bar{q}$ and λ similar to (2.3), i.e. ,

$$\frac{dW_1}{dr} = \lambda W_2 - \frac{W_1}{r} \ , \tag{3.3a}$$

$$\frac{d^2W_2}{dr^2} = -R\{2rW_1 + \lambda(1-r^2)W_2\} - \frac{1}{r}\frac{dW_2}{dr} - \lambda^2 W_2 - \lambda\bar{q} \ , \tag{3.3b}$$

$$\frac{d\bar{q}}{dr} = R(1-r^2)\lambda W_1 - \lambda\frac{dW_2}{dr} + \lambda^2 W_1 \ , \tag{3.3c}$$

with boundary conditions,

$$W_1(0) = \frac{dW_2}{dr}(0) = 0 \ \text{ and } \ W_1(1) = W_2(1) = 0 \ . \tag{3.3d}$$

By solving this system we may represent the solution to (2.1), for z far downstream, in the same way as for the channel flow problem i.e., by expressions similar to (2.4a) to (2.4d).

Similar to the channel flow problem we solved the system (3.3) numerically, using the solution to the system for $R = 0$ (the Stokes problem). These solutions may be found as

follows. When $R = 0$ the system (3.3) becomes,

$$\frac{dW_1}{dr} = \lambda W_2 - \frac{W_1}{r} \ , \tag{3.4a}$$

$$\frac{d^2W_2}{dr^2} = -\frac{1}{r}\frac{dW_2}{dr} - \lambda^2 W_2 - \lambda\bar{q} \ , \tag{3.4b}$$

$$\frac{d\bar{q}}{dr} = \lambda\frac{dW_2}{dr} + \lambda^2 W_1 \ , \tag{3.4c}$$

and boundary conditions (3.3d).

To find an explicit solution to this system differentiate (3.4c) with respect to r, multiply (3.4a) by λ^2 and (3.4b) by λ, then add the resulting expressions and using (3.4c) we obtain the following differential equation in $\bar{q}$,

$$\frac{d^2\bar{q}}{dr^2} + \frac{1}{r}\frac{d\bar{q}}{dr} + \lambda^2\bar{q} = 0 \ . \tag{3.5}$$

This equation has the particular solution,

$$\bar{q}(r) = J_0(\lambda r) \quad , \tag{3.6}$$

where J_0 is the Bessel function of first kind and order zero. This is the only solution, up to a multiplicative constant, which is finite when r is equal to zero. Substituting $\bar{q}(r)$ in (3.4b) we obtain the following differential equation in W_2,

$$\frac{d^2W_2}{dr^2} - \frac{1}{r}\frac{dW_2}{dr} - \lambda^2 W_2 = -\lambda J_0(\lambda r) \ , \tag{3.7a}$$

with boundary conditions,

$$\frac{dW_2}{dr}(0) = W_2(1) = 0 \ . \tag{3.7b}$$

This two-point boundary value problem has the following solution,

$$W_2(r) = \bar{b}J_0(\lambda r) - \frac{1}{2}rJ_1(\lambda r) \ , \tag{3.8}$$

where $\bar{b}$ is a constant to be determined by $W_2(1) = 0$. Using (3.4a) then W_1 is given as,

$$W_1(r) = \frac{1}{\lambda}\left[\frac{\lambda}{2} r J_0(\lambda r) - (1 - \lambda\bar{b}) J_1(\lambda r)\right] . \tag{3.9}$$

Since the solutions (3.8) and (3.9) need to satisfy the boundary conditions (3.3d) we have that,

$$\lambda W_1(1) = \frac{\lambda}{2} J_0(\lambda) - (1 - \lambda\bar{b}) J_1(\lambda) = 0 \quad , \tag{3.10a}$$

and

$$W_2(1) = \bar{b} J_0(\lambda) - \frac{1}{2} J_1(\lambda) = 0 . \tag{3.10b}$$

By eliminating $\bar{b}$ from these two equations we obtain,

$$\frac{1 \pm \sqrt{1 - \lambda^2}}{2\lambda} J_0(\lambda) - \frac{1}{2} J_1(\lambda) = 0 . \tag{3.11}$$

To solve this equation we observe that (3.11) can be transformed, after some algebraic manipulations, into the following non-linear equation

$$J_1(\lambda) J_0(\lambda) - \frac{\lambda}{2}(J_1^2(\lambda) + J_0^2(\lambda)) = 0 . \tag{3.12}$$

Lemma 3.1

All non-trivial solutions to the equation (3.12), that is those with $\lambda \neq 0$, are complex.

Proof

We make use of the following integral formula involving the Bessel functions.

$$\int x J_1^2(x) dx = \frac{1}{2} x^2 (J_0^2(x) + J_1^2(x)) - x J_0(x) J_1(x) . \tag{3.13}$$

This formula is obtained by integrating by parts Lommel's integral formula with $\alpha = 1$ (see [2], p 10). Then assuming that some λ satisfying (3.12) is real we get the contradiction that

$$0 < \int_0^\lambda xJ_1(x)^2dx = \frac{1}{2}\lambda^2(J_0^2(\lambda) + J_1^2(\lambda)) - \lambda J_0(\lambda)J_1(\lambda) = 0 \ ,$$

where we have assumed, without loss of generality, that $\lambda > 0$. This contradiction proves the lemma.

It is known that the Bessel functions have the following asymptotic representations,

$$J_0(\lambda) = (\frac{2}{\pi\lambda})^{1/2}\{\cos(\lambda - \frac{\pi}{4}) + \varepsilon_1(\lambda)\} \ , \tag{3.14a}$$

and

$$J_1(\lambda) = (\frac{2}{\pi\lambda})^{1/2}\{\sin(\lambda - \frac{\pi}{4}) + \varepsilon_2(\lambda)\} \ , \tag{3.14b}$$

where $\varepsilon_1(\lambda), \varepsilon_2(\lambda)$ can be regarded as small order terms for $Re(\lambda)$ large. Substituting these asymptotic expressions in (3.12), and neglecting $\varepsilon_1(\lambda)$ and $\varepsilon_2(\lambda)$, the λ's can be approximated by solutions of the following transcendental equation,

$$\cos(2\tilde{\lambda}) + \tilde{\lambda} = 0 \ . \tag{3.15}$$

Solutions of this equation can be used as initial iterates for a Newton method to obtain the roots of (3.12). We will present the detail of the computation in section 5.

Similar to the channel flow problem we used the solution to the Stokes problem to numerically solve the system (3.3) for arbitrary Reynolds numbers.

4. Method of Solution

In this section we discuss a method to solve the two-point boundary value problems for the systems (2.3), (3.3) and equation (2.5) and determine their eigenvalues. The method

can be described as follows, first the eigenvalue problem is transformed into an equivalent two-point boundary value first-order system of the form,

$$\vec{\mathbf{Z}}' = \mathbf{A}(\lambda, R, t)\vec{\mathbf{Z}} \quad , \quad a \leq t \leq b \ , \tag{4.1a}$$

$$\lambda' = 0 \ , \tag{4.1b}$$

with boundary conditions

$$\mathbf{B}_a \vec{\mathbf{Z}}(a) - \mathbf{B}_b \vec{\mathbf{Z}}(b) = 0 \ , \tag{4.1c}$$

$$z_k(b) = 1 \ , \tag{4.1d}$$

for some k , $1 \leq k \leq N$. Here $\vec{\mathbf{Z}} = (z_1, z_2, \cdots, z_N)$, and $\mathbf{A}, \mathbf{B_a}, \mathbf{B_b}$ are $N \times N$ matrices. The prime in (4.1a) indicates derivatives with respect to t. The condition (4.1d) has to be chosen in a way that is not in conflict with (4.1c). In the case of systems (2.3), (3.3) and equation (2.5) we used for condition (4.1d) $W_2'(1) = 1$, $W_1'(1) = 1$, and $W''(1) = 1$, respectively. To numerically solve the system (4.1) we used the two-point boundary value system code DVCPR from the IMSL library (this code is also known as PASVA3), it is a standard solver for first-order system of ordinary differential equations with conditions at two end points. It uses a variable step, with an automatic criterion to select a non-uniform grid, and a variable order of accuracy, with an excellent correspondence between the requested tolerance (tol) and the actual global error in the numerical solution, (see [10]). Since the systems considered are non-linear we used as initial iterate in PASVA3 the solution to the corresponding Stokes problem ($R = 0$) , which is known and for which the boundary conditions are satisfied, to generate solutions for arbitrary Reynolds numbers. This procedure is called continuation. The disadvantages of this method are that we have

to compute the eigenvalues one at a time, however the eigenvalues are computed together with their eigenfunctions. Also we can not compute the solution at a critical Reynolds number, i.e. a Reynolds number for which the solution to the system is not isolated, however it is possible to compute solutions for Reynolds numbers close to a critical value. By using this method we avoid the computation of spurious eigenvalues and also for high Reynolds numbers the solution can be computed in a very accurate manner.

5. Numerical Results

We now present some numerical results concerning the eigenvalue problems described in §2 and §3. For the channel flow problem our results are in full agreement with the previous computation of Bramley and Dennis [4], also we obtain the same answer by solving the system (4.1) corresponding to the primitive formulation, i.e., equations (2.3), and the system (4.1) corresponding to the stream function formulation, i.e., equation (2.5).

We present about the same number of results as in [4]. We present our results graphically concerning the behavior of these eigenvalues in the complex plane. We regard these eigenvalues as a function of Reynolds numbers in the form $\lambda(R) = \alpha(R) + i\beta(R)$ then we plotted the pairs $(\alpha(R), \beta(R))$ in the λ plane. We computed even and odd branches of eigenvalues and eigenvalues for axi-symmetric flow. Note that for the axi-symmetric flow there is no associated parity, such as even or odd, thus when we refer to an even or odd solution it means an even or odd solution to the channel problem.

For the channel flow problem we have found that these eigenvalues behave as follows (see figures 1.1a and 1.1b). Let $(\lambda_k(0), \vec{Z}_k(t))$ be an eigenpair solution for $R = 0$ where $| Re(\lambda_k) | < | Re(\lambda_{k+1}) |$, $k = 1, 2, \cdots$. For $0 \leq R < R^k_{c,0}$ $(\lambda_k(R), \vec{Z}_k(t))$ remains

complex, its conjugate also being an eigenpair solution. At $R = R^k_{c,0}$ this complex solution and its conjugate coalesce on the real axis and then, for R larger than $R^k_{c,0}$ they split into two branches of real solutions, one with increasing eigenvalues and the other with decreasing eigenvalues. For solutions with positive eigenvalues the increasing real solution $(\lambda_k(R), \vec{\mathbf{Z}}_k(t))$ coalesces with the decreasing real solution $(\lambda_{k+1}(R), \vec{\mathbf{Z}}_{k+1}(t))$ of the next branch at $R = R^k_{c,1}$ then, for larger values of R these two solutions split into a complex solution and its conjugate and again they remain complex for $R^k_{c,1} < R < R^k_{c,2}$, then at $R = R^k_{c,2}$ the complex solution and its conjugate again coalesce on the real axis and the cycle is repeated. Notice that $R^k_{c,m}$, $m = 1, 2, \cdots$, are the critical Reynolds numbers. There is an exception to this cycling phenomenon the first even branch of solution remains complex and approaches zero as R increases. Also the first real branch of even and odd decreasing eigenvalues approaches zero as R increases. None of the previous researchers have reported the second type of critical Reynolds number at which two real solutions coalesce and split into a complex solution and its conjugate. Only Wilson [12] pointed out that the positive increasing real eigenvalues approach a fixed value on the real axis. As can be seen from Figures 1.1a and 1.1b the cycles of the eigenvalues appear to be converging to a value on the real axis.

For eigenvalues with negative real part the situation is different, solutions with eigenvalues which decrease in magnitude either approach zero as R increases, but in a much slower rate than solutions with positive real eigenvalues, or they approach a fixed value. In these cases, the real eigenvalues increase or decrease very slowly that for the range of

the Reynolds number for which we made the computations, we did not observe the coalescence of real eigenvalues with negative real part.

We have found that for the axi-symmetric problem the solutions behave in the same manner as the odd solutions for the channel problem. That is, the solution are initially complex, the conjugate pairs coalesce on the real axis, the real solutions, with positive eigenvalues, coalesce, becoming complex, etc.

Figures 1.1*a* and 1.1*b* show the eigenvalues with positive real part in the λ plane for the first and second branches of even and odd eigenvalues, respectively, without including the complex conjugate. Also we do not include graphical displays for the axi-symmetric case since it is similar to the odd case. These graphics have been obtained for some range of R which are different for each curve. For the even eigenvalues the first curve, corresponding to the complex branch, was made for $0 \leq R \leq 10^6$ while the second branch for $0 \leq R \leq 100$. For odd eigenvalues the first curve corresponds to $0 \leq R \leq 90$, and the second for $0 \leq R \leq 100$. These differences arise because we wanted to illustrate the cycles for each branch. All the computations were done on the VAX 11/780 at the Mathematics Research Center at the University of Wisconsin-Madison.

Similar to [4] we present several tables which include the computation involving the eigenvalues. For $R = 0$, in the channel case, the eigenvalues may be computed by solving the transcendental equations (2.8*a*) and (2.8*b*) . Results concerning the solutions of these equations can be found in [6] and [11]. However, for the axi-symmetric case since there are not previous results we may start by computing the first few eigenvalues, with positive real part, which correspond to solutions of equation (3.12). As we said in §3 to solve equation

(3.12) we used solutions to (3.15) as initial iterates for the Newton method. At the same time to solve (3.15) we used the Newton method and as initial iterates the expressions,

$$\lambda_n = \gamma_n - \frac{\log(2\gamma_n)}{4\gamma_n} + \frac{i}{2}\log(2\gamma_n) \ , \tag{5.2}$$

with $\gamma_n = \frac{(2n+1)\pi}{2}$, $n = 1, 2\cdots$, since these expressions are asymptotic solutions to (3.15). In each case we have found that only two or three iterations were needed for the Newton method to find solutions to (3.12) and (3.15) with six decimal places of accuracy. The first several solution to the equation (3.12) are given in Table 1. Also we are including only those tables for which we have obtained results that are not reported in [4] (the complete set of tables regarding all the numerical results can be found in [1]), for example in Tables 2, 3, 4, 5, 6 we show real and complex eigenvalues for the axi-symmetric case, in Tables 8, 9 we give the critical Reynolds numbers, and eigenvalues, corresponding to graphics (1.1a) and (1.1b), in Tables 10, 11 we show the eigenvalues for high Reynolds number calculations.

As pointed out in [4], for large values of R the eigenvalues which have large modulus tend to be less accurate. Since the complex eigenvalues with negative real part have larger modulus than those with positive real part they may be a sensitive quantity in the computation. We computed the first branch of odd complex eigenvalues for $R \geq 10$ with different tolerances (the tolerance tol is a parameter in PASVA3 that controls the grow of the estimate errors). We have found that the accuracy of the eigenvalues increased considerably while we decreased the tolerance, e.g., for $tol = 10^{-6}$ and 10^{-8} the eigenvalues agree up to six decimal places for Reynolds numbers in the range $10 \leq R \leq 50$ and they have at least three decimals places in common for $R > 50$, while for $tol = 10^{-8}$ and

10^{-10} the eigenvalues agree up to six decimal places for Reynolds numbers in the range $10 \leq R \leq 10000$. We present the corresponding results in Table 7.

6. Eigenvalues at Critical Reynolds Numbers

As we pointed out before this method fails to compute eigenvalues at critical Reynolds numbers. In [4] Bramley and Dennis computed some eigenvalues, that they suggested occur at critical Reynolds numbers, e.g. at $R = 6.3$ they computed the eigenvalue $\lambda \approx 6.0$, which corresponds to the first critical eigenvalues of the second odd branch (see figure 1.1b). We have found, according to our computation, that $R = 6.3$ is not exactly a critical Reynolds number but it is very close to it, i.e., we computed two eigenvalues for that R, one corresponding to the increasing solution with $\lambda = 6.01267$ (which, apparently, agrees with the value given by Bramley and Dennis) and the other corresponding to the decreasing solution i.e. $\lambda = 5.85346$. A similar situation happens for a critical eigenvalue reported in [7] by Gillis and Brandt. They computed the eigenvalue $\lambda \approx 2.632$, presumably a critical one, at $R = 8.461$. We compute eigenvalues $\lambda = 2.62085$ and $\lambda = 2.63875$ at that Reynolds number. We believe that it is difficult to determine exactly a critical Reynolds number and its critical eigenvalue. Here we introduce a device, which may be useful, to compute better approximations to critical Reynolds numbers and their eigenvalues. First we assume that the eigenvalues, near a critical Reynolds number, may be represented by the following formula,

$$\lambda = \lambda_c + \bar{a}\sqrt{R - R_c} , \tag{6.1}$$

where R_c is the critical Reynolds number, λ_c the critical eigenvalue and $\bar{a}$ is a constant. For Reynolds numbers R_1, R_2, R_3, near R_c we can rewrite (6.1), after some algebraic

manipulation, as

$$R_i = R_1 + A(\lambda_i - \lambda_1) + B(\lambda_i - \lambda_1)^2 \ , \tag{6.2}$$

with A and B given by $A = (1/\bar{a})^2$ and $B = (1/\bar{a})^2(2\lambda_1 - 2\lambda_c)$ respectively, and $i = 2,3$. Then, using the values of R_i, λ_i , we solved the system (6.2) in terms of A, B . We can compute λ_c by the formula

$$\lambda_c = \lambda_1 - \frac{A}{2B} \quad , \tag{6.3}$$

while R_c is obtained using (6.1). This approach seems to work well, whenever the eigenvalues are close to λ_c. In Tables 8 and 9 we show all the critical eigenvalues, corresponding to Figures 1.1a and 1.1b, computed by this approach.

7. Asymptotic Behavior for Large R

It is interesting to see the behavior of these eigenvalues for high Reynolds numbers. By high Reynolds number we mean that the system (4.1) becomes a stiff system of ordinary differential equations. In [12], Wilson developed a theory based on a singular perturbation analysis to investigate these eigenvalues. He considered equation (2.5), corresponding to a channel flow problem in the stream function formulation. For high Reynolds numbers Wilson found that the eigenvalues approaching zero downstream fell in two categories, one in which the eigenvalues approached zero as $O(R^{-1})$ and in the other as $O(R^{-\frac{1}{2}})$. From Tables 5 we see that the eigenvalues behaving like $O(R^{-1})$ correspond to positive real decreasing eigenvalues, which according to Wilson, these eigenvalues may be written as $\lambda = \lambda_0 R^{-1} + O(R^{-3})$ with λ_0 being a constant. By neglecting smaller terms we have that $\lambda_0 \cong \lambda R$. Then it is possible to use the numerical data to compute λ_0. We did that

for $R = 10^5$ and the λ_0's that we found agree with those given by Wilson, (see [12], Table 3).

The second type of eigenvalues approaching zero downstream, according to our computation, are the even complex eigenvalues (first branch). By assuming that these eigenvalues can be written as

$$\lambda = \lambda_0 R^{-\frac{1}{7}} + \cdots \quad , \tag{7.1}$$

we have that for R large enough, neglecting smaller perturbations, we may compute the λ_0's , using the numerical data, by $\lambda_0 \approx \lambda R^{\frac{1}{7}}$. In Table 10 we present the corresponding results.

Notice that λ_0 tends to a constant as R increases, it seems to be that R has to be quite large before neglecting any smaller perturbation in the asymptotic expression for λ. Unfortunately, Wilson did not mention how to obtain the smaller order terms in (7.1). Using the numerical data we may attempt to compute the second lower order term in (7.1) by assuming that λ can be expanded as

$$\lambda = bR^{-\frac{1}{7}} + cR^{-d} + \cdots \quad , \tag{7.2}$$

with b, c, d constants to be determined. We have then that for four consecutive eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4 , at Reynolds numbers of the form $R_k = 10R_{k-1}$, the following formula

$$10^{-\frac{1}{7}}\alpha_2 - \alpha_3 = Re(c)10^{-d}(10^{-\frac{1}{7}} - 10^{-d})R_1^{-d} \; . \tag{7.3a}$$

and

$$10^{-\frac{1}{7}}\alpha_3 - \alpha_4 = Re(c)10^{-2d}(10^{-\frac{1}{7}} - 10^{-d})R_1^{-d} \; . \tag{7.3b}$$

where $\lambda_k = \alpha_k + i\beta_k$, i.e., we have that d is given by

$$d = \log\left(\frac{10^{-\frac{1}{7}}\alpha_2 - \alpha_3}{10^{-\frac{1}{7}}\alpha_3 - \alpha_4}\right) \Big/ \log\ 10\ , \tag{7.4}$$

while b can be determined, using again (7.2), by

$$b = \frac{10^{-d}\lambda_3 - \lambda_4}{(10^{-d} - 10^{-\frac{1}{7}})R_1^{-\frac{1}{7}}}\ . \tag{7.5}$$

and c is determined using (7.2) with λ replaced by λ_4 .

We found the following values for b, c and d, $b = 1.6400 - 0.7813i$, $c = -0.1655 - 0.0435i$ and $d = 0.34676$. We used λ_k corresponding to $R_k = 10^3, 10^4, 10^5, 10^6$. Therefore by substituting these values in (7.2) it is possible to obtain a better asymptotic prediction. Notice that d is approximately $\frac{5}{14}$ i.e., the eigenvalues behave, approximately, as

$$\lambda = (1.64 + 0.7813i)R^{-\frac{1}{7}} + O(R^{-\frac{5}{14}})\ . \tag{7.5}$$

In Table 11 we show a comparation between the numerical results of the eigenvalue calculations and the asymptotic results using formula (7.2). The agreement is satisfactory, notice that for the Reynolds numbers used to derive formula (7.2) the agreements are better.

8. Conclusion

The eigenvalues governing the rate of decay of perturbations of Poiseuille flow have been studied. The procedure to compute the eigenvalues is based upon expressing the eigenvalue system as a two-point boundary value problem. The eigenvalues are computed one at a time jointly with the eigenfunctions, the results presented are in agreement with previous computations [4] and [12]. For high Reynolds numbers we show that the first

branch of even eigenvalues has an asymptotic representation of the form (7.5). We think that our procedure is simpler, more direct and efficient than previous methods and can be used with relative ease on a computer. Due to the high accuracy obtained on using this method we strongly recommend it for computations involving eigenvalue systems of ordinary differential equations.

Acknowledgements

I would like to thank Professor John C. Strikwerda for his valuable advice and the encouragement I have received during the preparation of this research.

This paper is a portion of my Ph.D dissertation, carried out at the University of Wisconsin-Madison. The final report was written while I was visiting the Center for the Mathematical Sciences, University of Wisconsin-Madison.

This work was supported in part by the United States Army under Contract No. DAAG29-80-C-0041 and in part by Consejo de Desarrollo Científico y Humanístico, UCV, Caracas.

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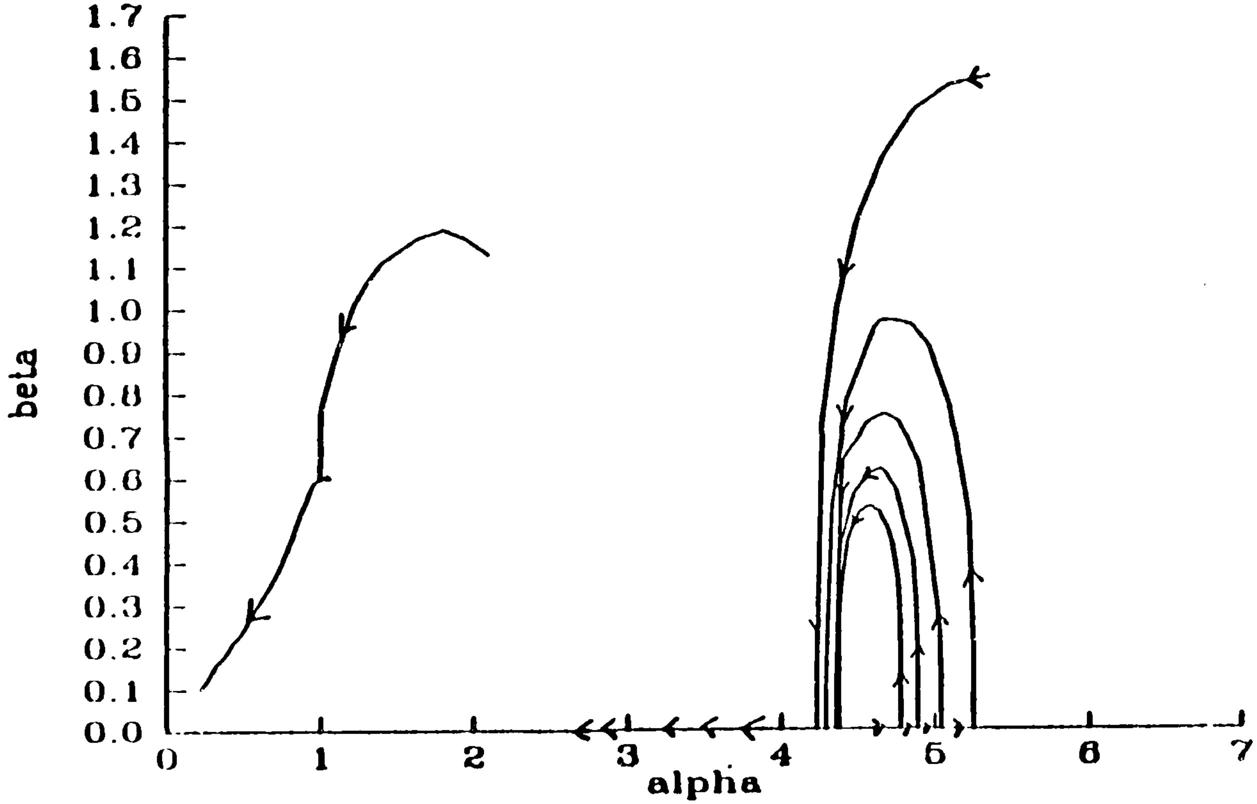

FIGURE 1.1a : Even eigenvalues in the λ plane.

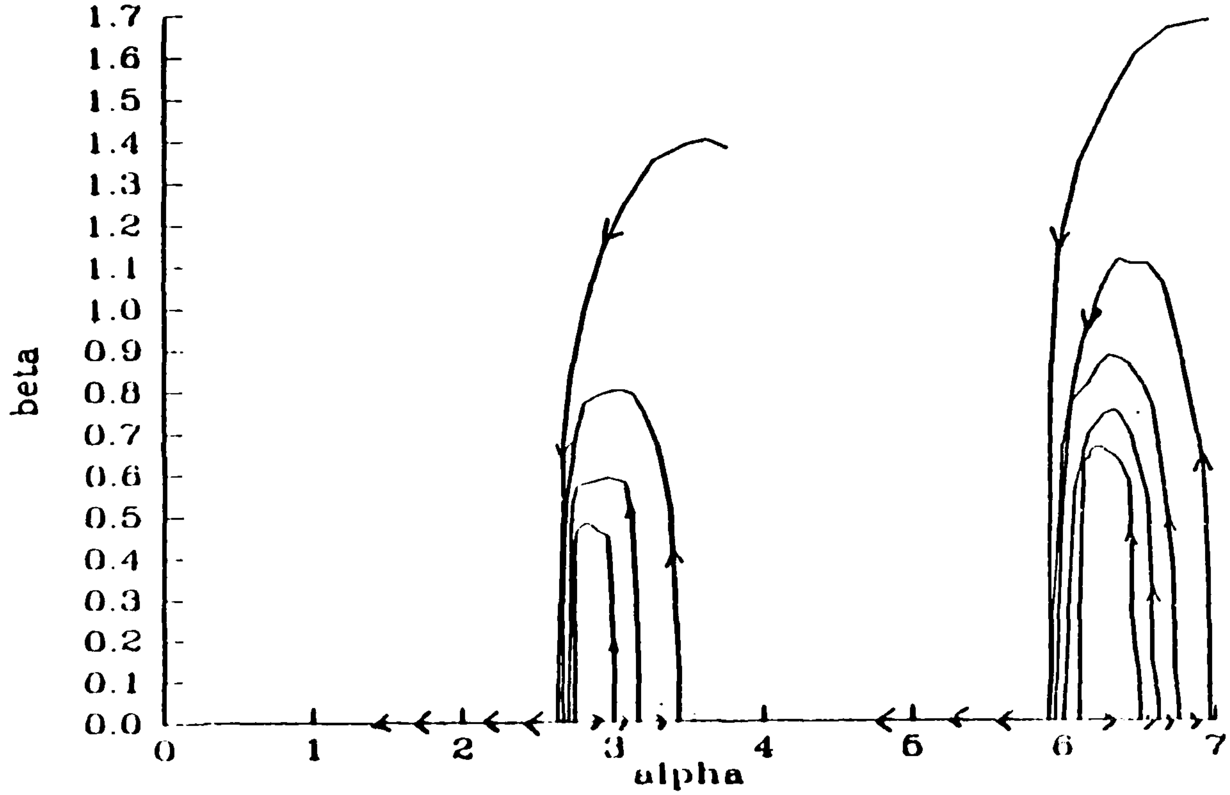

FIGURE 1.1b : Odd eigenvalues in the λ plane.

TABLE 1

Solutions to equation (3.12)

λ
4.46630−1.46747i
7.69410−1.72697i
10.87457−1.89494i
14.03889−2.02006i
17.19557−2.11995i
20.34797−2.20312i
23.49772−2.27441i
26.64572−2.33678i
29.79248−2.39223i
32.93833−2.44214i

TABLE 2

Axi-symetric complex eigenvalues with positive real part for $0 \leq R \leq 100$

R	λ	λ	λ
0	4.46620−1.46747i	7.69410−1.72697i	10.87457−1.89494i
1	4.29200−1.42646i	7.52551−1.69447i	10.70765−1.86725i
5	3.76255−1.06197i	6.95388−1.31088i	10.11519−1.47824i
15	4.24793−0.78668i	7.31772−1.10920i	10.40317−1.28557i
25	3.70720−0.50937i	7.77552−0.31462i	10.82173−0.95050i
50	3.79197−0.45240i	7.44769−0.69724i	10.31659−0.90353i
100	4.17041−0.27261i	7.48688−0.33362i	10.22035−0.64556i

TABLE 3

Complex eigenvalues with smallest positive real part for $R > 100$

R	λ^e(even)	λ^o(odd)	λ(axi-symm)
250	2.78306−0.27633i	0.72120−0.37930i	3.80091−0.06650i
500	2.71137−0.15264i	0.65637−0.33642i	3.97900−0.17319i
1000	2.70753−0.15583i	0.59650−0.29956i	3.86180−0.15698i
2500	2.67659−0.12999i	0.38104−0.18322i	3.50706−0.09679i
5000	2.59490−0.08062i	0.34571−0.16572i	

TABLE 4

Axi-symmetric complex eigenvalues with negative real part for $0 \le R \le 10$

R	λ	λ	λ
0.25	-4.5126−1.4738i	-7.7378−1.7309i	-10.9174−1.8976i
0.50	-4.5601−1.4784i	-7.7821−1.7331i	-10.9607−1.8986i
1.00	-4.6584−1.4823i	-7.8727−1.7318i	-11.0486−1.8949i
2.50	-4.9823−1.4417i	-8.1593−1.6744i	-11.3231−1.8331i
5.00	-5.6481−1.0772i	-8.6844−1.2420i	-11.8148−1.4338i
10.00	-8.3161−1.5370i	-11.4575−1.4277i	-14.4887−1.5590i

TABLE 5

Real positive axi-symmetric eigenvalues for $R \leq 5000$

R	λ	λ	λ	λ	λ
10	2.84882				
25	1.23820				
50	0.63541	1.86985			
100	0.32000	0.95032	1.89301	3.21182	
250	0.12827	0.38230	0.76379	1.27228	1.90837
500	0.06415	0.19134	0.38250	0.63760	0.95652
1000	0.03208	0.09568	0.19133	0.31900	0.47867
5000	0.00642	0.01914	0.03827	0.06381	0.09576

TABLE 6

Real negative axi-symmetric eigenvalues for $R \leq 10^5$

R	λ	λ	λ
10	-4.94606		
25	-4.49291	-8.13980	-12.42763
50	-4.31816	-7.77838	-11.24747
100	-4.19980	-7.57708	-10.92172
250	-4.09219	-7.40731	-10.68291
500	-4.03427	-7.31888	-10.56491
1000	-3.99010	-7.25228	-10.47775
2500	-3.94679	-7.18740	-10.39379
5000	-3.92235	-7.15090	-10.34684
10000	-3.90323	-7.12238	-10.31023
50000	-3.87316	-7.07752	-10.25277
100000	-3.86452	-7.06463	-10.23627

TABLE 7

Odd complex eigenvalues with negative real part for $R \geq 10$

R	λ^o (tol=10^{-6})	λ^o (tol$\geq 10^{-8}$)
10	-10.458682−1.694815i	-10.458682−1.694815i
25	-16.941646−2.763071i	-16.941646−2.763071i
50	-24.288883−4.070337i	-24.288883−4.070337i
100	-34.664217−5.966789i	-34.664089−5.966881i
250	-55.230448−9.710819i	-55.230451−9.710823i
500	-113.589146−11.159176i	-113.589146−11.159176i
1000	-111.168228−19.879302i	-111.168232−19.879303i
2500	-176.178035−31.692081i	-176.178035−31.692081i
5000	-362.979226−36.296213i	-362.979211−36.296142i
10000	-352.923824−63.768959i	-352.923830−63.768960i

TABLE 8

Critical Reynolds numbers and critical odd eigenvalues

$R^k_{c,0}$	λ^o_c	$R^k_{c,1}$	λ^o_c
8.4606	2.63126	6.2983	5.89598
9.1576	3.42632	6.8691	6.97998
25.6282	2.67267	17.1490	5.91199
26.3405	3.15830	18.0316	6.76558
50.8847	2.70372	31.6603	5.95620
51.3489	2.95271	32.6501	6.63307
84.1917	2.74161	49.3053	6.00915
		50.7706	6.51354
		71.4975	6.19109

TABLE 9

Critical Reynolds numbers and critical even eigenvalues

$R_{c,0}^{k}$	λ_c^e
6.8846	4.23405
7.5805	5.25272
19.5660	4.29246
20.5465	5.04126
37.1756	4.36997
38.2524	4.87873
59.8274	4.37883
60.7830	4.81713
87.3756	4.38935

TABLE 10

Even complex eigenvalues for high Reynolds numbers

R	λ^e	λ_0
1000	0.59650−0.29956i	1.60023−0.80362i
10000	0.43318−0.20964i	1.61472−0.78146i
100000	0.31358−0.15003i	1.62420−0.77709i
1000000	0.22651−0.10819i	1.63013−0.77865i

TABLE 11

Numerical vs Asymptotic Eigenvalues at high Reynolds numbers

R	λ^e (numer)	λ^e (asympt)
1000	0.59650−0.29956i	0.59625−0.28725i
10000	0.43317−0.20964i	0.43317−0.20780i
50000	0.34571−0.16572i	0.34572−0.16552i
100000	0.31358−0.15003i	0.31358−0.15004i
500000	0.24986−0.11934i	0.24986−0.11950i
750000	0.23592−0.11269i	0.23592−0.11271i
1000000	0.22651−0.10819i	0.22651−0.10819i

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